JOURNAL OF COMPUTATIONAL PHYSICS 113, 309–319 (1994)

A Conservative Difference Scheme for the Zakharov Equations*

QIANSHUN CHANG† AND HONG JIANG

Department of Computer Science, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

Received June 18, 1993; revised January 12, 1994

A new conservative difference scheme is presented for the periodic initial-value problem of Zakharov equations. The scheme can be implicit or semi-explicit, depending on the choice of a parameter. The discretization of the initial condition is of second-order accuracy, which is consistent with the accuracy of the scheme. On the basis of a priori estimates and an inequality about norms, convergence of the difference solutions is proved in the energy norm. Numerical experiments with the schemes are done for several test cases. Computational results demonstrate that the new semi-explicit scheme with a new initial approximation is more accurate and computationally efficient.

1. INTRODUCTION

The Zakharov equations [9]

$$iE_t + E_{xx} - \mathcal{N}E = 0, \tag{1.1}$$

$$\mathcal{N}_{tt} - (\mathcal{N} + |E|^2)_{xx} = 0 \tag{1.2}$$

describe the propagation of Langmuir waves in plasmas. Here E is the slowly varying envelope of highly oscillatory electric field, and $\mathcal{N}$ denotes the fluctuation in the ion-density about its equilibrium value.

In [11], the global existence of a weak solution for the Zakharov equations is proved, and the existence and uniqueness of a smooth solution in one dimension are obtained, provided that smooth initial data are prescribed.

Numerical methods for the Zakharov equations were considered in [5, 6, 9]. A spectral method was used to compute solitary waves and the interaction of two colliding solitary waves in [9]. In [5, 6], Glassey considered an energy-preserving implicit difference scheme for the equations and proved its convergence.

In this paper, we consider the solution of the Zakharov equations with smooth initial values and periodic conditions; i.e., the initial conditions are supplied,

$$E(x,0) = E^0(x), \qquad \mathcal{N}(x,0) = \mathcal{N}^0(x), \\ \mathcal{N}_t(x,0) = \mathcal{N}^1(x), \tag{1.3}$$

where $E^0(x)$, $\mathcal{N}^0(x)$, and $\mathcal{N}^1(x)$ are periodic functions with period L; $\mathcal{N}^1(x)$ satisfies the compatibility condition

$$\int_0^L \mathcal{N}^1(x)\,dx = 0. \tag{1.4}$$

The initial value problem (1.1) to (1.4) possesses two conservative quantities,

$$\int_0^L |E(x,t)|^2\,dx = \text{const} \tag{1.5}$$

$$\int_0^L \left(|E_x|^2 + \tfrac{1}{2}(v^2 + \mathcal{N}^2) + \mathcal{N}\,|E|^2\right) dx = \text{const}, \tag{1.6}$$

where $v = -u_x$ and $u_{xx} = \mathcal{N}_t$.

We propose a new conservative difference scheme which involves a parameter θ, $0 \leqslant \theta \leqslant \frac{1}{2}$. The scheme conserves the two invariants (1.5) and (1.6) for any θ, $0 \leqslant \theta \leqslant \frac{1}{2}$. When $\theta = \frac{1}{2}$, the new scheme is identical to Glassey's scheme in [5, 6]. For $\theta = 0$, the new scheme is semi-explicit, explicit in $\mathcal{N}$, but implicit in E. Numerical experiments demonstrate that the new scheme with $\theta = 0$ is more accurate and efficient compared to $\theta = \frac{1}{2}$. For example, when the computational error is required to be less than 0.1 over one period of a solitary wave, the scheme with $\theta = \frac{1}{2}$ requires a CPU time about six times higher than that with $\theta = 0$.

We will also present a discretization of the initial condition with truncation error $O(\tau^2)$, instead of the simple initial approximation with error $O(\tau)$. The truncation error of $O(\tau^2)$ is consistent with the difference scheme with error $O(h^2 + \tau^2)$.

A priori estimates for the solution of the difference scheme

* This work was supported by the National Science and Engineering Research Council of Canada.

† Permanent address is the Institute of Applied Mathematics, The Chinese Academy of Sciences, Beijing 100080, China.

0021-9991/94 $6.00

will be made, and a useful inequality about norms will be obtained. Convergence of the difference solution in the energy norm will be proved based on the inequality. The result that the scheme is convergent for all $\theta \in [0, \frac{1}{2}]$ is significant because, although the convergence of the implicit scheme ($\theta = \frac{1}{2}$) has been known [6], the convergence of the more general case ($\theta \neq \frac{1}{2}$) is more difficult to prove and requires a different approach.

Various numerical experiments will be performed. Numerical solutions will be compared with an analytical solution of Zakharov equations, and the collision of two solitary waves will also be considered. The computational results will be discussed and compared in detail.

In Section 2, we describe the difference scheme. In Section 3, we make some a priori estimates and prove the inequality about norms and the convergence of the new scheme with θ, $0 \leqslant \theta \leqslant \frac{1}{2}$. Finally, various numerical results will be discussed in Section 4.

2. NUMERICAL METHOD

In this section, we describe the numerical method for the Zakharov equations. As usual, the following notations are used:

$$x_j = jh, \qquad t^n = n\tau, \qquad 0 \leqslant j \leqslant J = \left[\frac{L}{h}\right],$$

$$n = 0, 1, 2, ..., \left[\frac{T}{\tau}\right]$$

$$E_j^n \sim E(x_j, t^n), \qquad \mathcal{N}_j^n \sim \mathcal{N}(x_j, t^n),$$

$$(W_j^n)_x = \frac{1}{h}(W_{j+1}^n - W_j^n), \qquad (W_j^n)_{\bar{x}} = \frac{1}{h}(W_j^n - W_{j-1}^n),$$

$$(W_j^n)_t = \frac{1}{h}(W_j^{n+1} - W_j^n), \qquad (W_j^n)_{\bar{t}} = \frac{1}{h}(W_j^n - W_j^{n-1}),$$

$$\|W^n\|_2^2 = h\sum_{j=1}^{J} |W_j^n|^2, \qquad \|W^n\|_\infty = \sup_{1 \leqslant j \leqslant J} |W_j^n|,$$

$$\gamma = \frac{\tau}{h}, \qquad \beta = \frac{\tau}{h^2}.$$

Thus our scheme is written as

$$i(E_j^{n+1})_{\bar{t}} + \tfrac{1}{2}((E_j^{n+1})_{x\bar{x}} + (E_j^n)_{\bar{x}x}) = \tfrac{1}{4}(\mathcal{N}_j^{n+1} + \mathcal{N}_j^n)(E_j^{n+1} + E_j^n), \qquad 1 \leqslant j \leqslant J, \tag{2.1}$$

$$(\mathcal{N}_j^n)_{t\bar{t}} - (1-2\theta)(\mathcal{N}_j^n)_{x\bar{x}} - \theta((\mathcal{N}_j^{n+1})_{x\bar{x}} + (\mathcal{N}_j^{n-1})_{\bar{x}x}) = (|E_j^n|^2)_{x\bar{x}}, \qquad 0 \leqslant \theta \leqslant \tfrac{1}{2}, \tag{2.2}$$

where E_j^n, $\mathcal{N}_j^n$ are J-periodic mesh functions, i.e.,

$$E_j^n = E_{j+J}^n, \qquad \mathcal{N}_j^n = \mathcal{N}_{j+J}^n. \tag{2.3}$$

The initial data are approximated as

$$E_j^0 = E^0(x_j), \tag{2.4}$$

$$\mathcal{N}_j^0 = \mathcal{N}^0(x_j), \tag{2.5}$$

$$\mathcal{N}_j^1 = \mathcal{N}_j^0 + \tau\mathcal{N}^1(x_j), \tag{2.6}$$

or

$$\mathcal{N}_j^1 = \mathcal{N}_j^0 + \tau\mathcal{N}^1(x_j) + \frac{\tau^2}{2}[(\mathcal{N}_j^0)_{x\bar{x}} + (|E_j^0|^2)_{x\bar{x}}]. \tag{2.7}$$

The scheme (2.1)–(2.2) can be rewritten in the following forms which are convenient for numerical computations:

$$\frac{\beta}{2}E_{j+1}^{n+1} + \left[i - \beta - \frac{\tau}{4}(\mathcal{N}_j^n + \mathcal{N}_j^{n+1})\right]E_j^{n+1} + \frac{\beta}{2}E_{j-1}^{n+1} = iE_j^n - \frac{\beta}{2}(E_{j+1}^n - 2E_j^n + E_{j-1}^n) + \frac{\tau}{4}(\mathcal{N}_j^n + \mathcal{N}_j^{n+1})E_j^n, \tag{2.8}$$

$$-\gamma^2\theta\mathcal{N}_{j+1}^{n+1} + (1 + 2\gamma^2\theta)\mathcal{N}_j^{n+1} - \gamma^2\theta\mathcal{N}_{j-1}^{n+1} = 2\mathcal{N}_j^n - \mathcal{N}_j^{n-1} + (1-2\theta)\gamma^2(\mathcal{N}_{j+1}^n - 2\mathcal{N}_j^n + \mathcal{N}_{j-1}^n) + \gamma^2\theta(\mathcal{N}_{j+1}^{n-1} - 2\mathcal{N}_j^{n-1} + \mathcal{N}_{j-1}^{n-1}) + \gamma^2(|E_{j+1}^n|^2 - 2|E_j^n|^2 + |E_{j-1}^n|^2). \tag{2.9}$$

In computations, E_j^0, $\mathcal{N}_j^0$, and $\mathcal{N}_j^1$ are obtained by initial data (2.4)–(2.7). E_j^1 is computed by (2.8) with $n = 0$. Then $\mathcal{N}_j^{n+1}$ is computed by (2.9) and E_j^{n+1} is computed by (2.8). The procedure is repeated until $n + 1 = [T/\tau]$.

We note that Eq. (2.8) is implicit and a periodic tridiagonal system of equations is involved. For $\theta = \frac{1}{2}$, Eq. (2.9) is also implicit and a periodic tridiagonal system of equations needs to be solved again. The scheme (2.8)–(2.9) with $\theta = \frac{1}{2}$ has been studied by Glassey in [5, 6]. However, when $\theta = 0$, the scheme (2.9) for $\mathcal{N}$ is explicit and no tridiagonal system needs to be solved. For this reason, this scheme is said to be semi-explicit. The main goal in this paper is to study the scheme (2.8)–(2.9) with $0 \leqslant \theta < \frac{1}{2}$.

3. CONVERGENCE OF DIFFERENCE SCHEME

In this section, we consider the convergence of difference scheme (2.1)–(2.7). We will assume that $\gamma = (\tau/h) \leqslant C$ and use C as a general constant, which may have different values in different occurrences.

Following the notations used by Glassey in [6], we define $\{U_j^{n+1/2}\}$ as the solution of

$$(U_j^{n+1/2})_{x\bar{x}} = (\mathcal{N}_j^n)_t, \qquad 1 \leqslant j \leqslant J-1, \tag{3.1}$$

$$U_0^{n+1/2} = U_J^{n+1/2} = 0. \tag{3.2}$$

LEMMA 1. *The difference problem* (2.1)–(2.7) *admits the following two invariants*:

$$P_h^n = \|E^n\|_2^2$$

and

$$H_h^{n+1/2} = \|E_x^{n+1}\|_2^2 + \|E_x^n\|_2^2 + \|U_x^{n+1/2}\|_2^2 + (1-2\theta)\,h \sum_{j=1}^{J} \mathcal{N}_j^{n+1}\cdot\mathcal{N}_j^n + \theta(\|\mathcal{N}^{n+1}\|_2^2 + \|\mathcal{N}^n\|_2^2) + \tfrac{1}{2}h \sum_{j=1}^{J} (\mathcal{N}_j^{n+1}+\mathcal{N}_j^n)(|E_j^{n+1}|^2 + |E_j^n|^2).$$

Proof. We first multiply both sides of (2.1) by $(E_j^{n+1}+E_j^n)$ and sum over j. Then taking the imaginary part of the result gives us the invariant P_h^n.

Computing the inner product of (2.1) with $\tau(E_j^{n+1})_{\bar{t}}$ and taking the real part we have

$$-(\|E_x^{n+1}\|_2^2 - \|E_x^n\|_2^2) = \tfrac{1}{2}h \sum_{j=1}^{J} (\mathcal{N}_j^{n+1}+\mathcal{N}_j^n)(|E_j^{n+1}|^2 - |E_j^n|^2). \quad (3.3)$$

Next, we compute the inner product of (2.2) with $(U_j^{n+1/2}+U_j^{n-1/2})$ and, after using (3.1), we obtain

$$((\mathcal{N}_j^n)_{t\bar{t}},\, U_j^{n+1/2}+U_j^{n-1/2}) - (1-2\theta)(\mathcal{N}_j^n,\, (\mathcal{N}_j^n)_t + (\mathcal{N}_j^{n-1})_t) - \theta(\mathcal{N}_j^{n+1}+\mathcal{N}_j^{n-1},\, (\mathcal{N}_j^n)_t + (\mathcal{N}_j^{n-1})_t) = (|E_j^n|^2,\, (\mathcal{N}_j^n)_t + (\mathcal{N}_j^{n-1})_t),$$

which is equivalent to

$$\|U_x^{n+1/2}\|_2^2 - \|U_x^{n-1/2}\|_2^2 + (1-2\theta)\,h \sum_{j=1}^{J} \mathcal{N}_j^{n+1}\cdot\mathcal{N}_j^n - (1-2\theta)\,h \sum_{j=1}^{J} \mathcal{N}_j^n\cdot\mathcal{N}_j^{n-1} + \theta(\|\mathcal{N}^{n+1}\|_2^2 - \|\mathcal{N}^{n-1}\|_2^2) + h \sum_{j=1}^{J} |E_j^n|^2\cdot(\mathcal{N}_j^{n+1}-\mathcal{N}_j^{n-1}) = 0, \quad (3.4)$$

where (3.1) is used to reduce the first term.

It follows from (3.3) that

$$-(\|E_x^{n+1}\|_2^2 - \|E_x^{n-1}\|_2^2) = \tfrac{1}{2}h \sum_{j=1}^{J} (\mathcal{N}_j^{n+1}+\mathcal{N}_j^n)(|E_j^{n+1}|^2 + |E_j^n|^2) - \tfrac{1}{2}h \sum_{j=1}^{J} (\mathcal{N}_j^n+\mathcal{N}_j^{n-1})(|E_j^n|^2 + |E_j^{n-1}|^2) - \tfrac{1}{2}h \sum_{j=1}^{J} |E_j^n|^2(\mathcal{N}_j^{n+1}-\mathcal{N}_j^{n-1}). \quad (3.5)$$

Combining (3.5) with (3.4) yields

$$H_h^{n+1/2} = H_h^{n-1/2} = \text{const}.$$

LEMMA 2. *Let* $\gamma = (\tau/h) < \sqrt{1/(1-2\theta)}$, $0 \leqslant \theta \leqslant \frac{1}{2}$. *If we define* $C_1 = (2+(1-2\theta)\gamma^2)/(2-(1-2\theta)\gamma^2)$, *then the inequality*

$$R_\tau \leqslant C_1 Q_\tau$$

holds, where

$$Q_\tau = \|U_x^{n+1/2}\|_2^2 + (1-2\theta)\,h \sum_{j=1}^{J} \mathcal{N}_j^{n+1}\cdot\mathcal{N}_j^n,$$

$$R_\tau = \|U_x^{n+1/2}\|_2^2 + \tfrac{1}{2}(1-2\theta)(\|\mathcal{N}^{n+1}\|_2^2 + \|\mathcal{N}^n\|_2^2).$$

Proof. Let $(W_j^n)_t = U_j^{n+1/2}$ and $W_j^0 = 0$, then $W_0^n = W_J^n = 0$ and $\mathcal{N}_j^n = (W_j^n)_{x\bar{x}}$. Thus, we have

$$Q_\tau = h \sum_{j=1}^{J} [(W_j^n)_{xt}]^2 + (1-2\theta)\,h \sum_{j=1}^{J} (W_j^{n+1})_{x\bar{x}}\cdot(W_j^n)_{x\bar{x}}$$

$$R_\tau = h \sum_{j=1}^{J} [(W_j^n)_{xt}]^2 + \tfrac{1}{2}(1-2\theta)\,h \times \sum_{j=1}^{J} [(\mathcal{N}_j^{n+1})^2 + (\mathcal{N}_j^n)^2].$$

We use the following notations:

$$DW_j^n \equiv (U_j^n)_x, \qquad D^2W_j^n \equiv (U_j^n)_{x\bar{x}},$$

$$Q_D \equiv \begin{bmatrix} -\tau^{-2}D^2, & \tau^{-2}D^2 + \frac{1}{2}(1-2\theta)D^4 \\ \tau^{-2}D^2 + \frac{1}{2}(1-2\theta)D^4, & -\tau^{-2}D^2 \end{bmatrix},$$

$$R_D \equiv \begin{bmatrix} -\tau^{-2}D^2 + \frac{1}{2}(1-2\theta)D^4, & \tau^{-2}D^2 \\ \tau^{-2}D^2, & -\tau^{-2}D^2 + \frac{1}{2}(1-2\theta)D^4 \end{bmatrix}.$$

It is easy to verify that

$$Q_\tau = h \sum_{j=1}^{J} (W_j^{n+1}, W_j^n) \times \begin{bmatrix} -\tau^{-2}D^2, & \tau^{-2}D^2 + \frac{1}{2}(1-2\theta) D^4 \\ \tau^{-2}D^2 + \frac{1}{2}(1-2\theta) D^4, & -\tau^{-2}D^2 \end{bmatrix} \times \begin{pmatrix} W_j^{n+1} \\ W_j^n \end{pmatrix}$$

$$= h \sum_{j=1}^{J} (W_j^{n+1}, W_j^n) Q_D \cdot \begin{pmatrix} W_j^{n+1} \\ W_j^n \end{pmatrix},$$

and

$$R_\tau = h \sum_{j=1}^{J} (W_j^{n+1}, W_j^n) R_D \cdot \begin{pmatrix} W_j^{n+1} \\ W_j^n \end{pmatrix}.$$

Assume that (Y_1, Y_2) is an eigenfunction associated with the eigenvalue λ of Q_D, then

$$-\tau^{-2}D^2 Y_1 + \tau^{-2}D^2 Y_2 + \tfrac{1}{2}(1-2\theta) D^4 Y_2 = \lambda Y_1,$$
$$\tau^{-2}D^2 Y_1 + \tfrac{1}{2}(1-2\theta) D^4 Y_1 - \tau^{-2}D^2 Y_2 = \lambda Y_2.$$

By adding and subtracting these equations, we obtain

$$\tfrac{1}{2}(1-2\theta) D^4(Y_1 + Y_2) = \lambda(Y_1 + Y_2), \tag{3.6}$$

$$-2\tau^{-2}D^2(Y_1 - Y_2) - \tfrac{1}{2}(1-2\theta) D^4(Y_1 - Y_2) = \lambda(Y_1 - Y_2). \tag{3.7}$$

If we look for an eigenfunction with $Y_1 = Y_2 = Y$, then (3.7) always holds and (3.6) implies that Y is an eigenfunction of the operator $\frac{1}{2}(1-2\theta) D^4$ with eigenvalue $\frac{1}{2}(1-2\theta)\mu_4$, where μ_4 is the eigenvalue of D^4. This provides J eigenvalues of Q_D. On the other hand, if we seek eigenfunctions with $Y_1 = -Y_2 = Y$, (3.6) holds and (3.7) implies that eigenvalue λ is of the form $-2\tau^{-2}\mu_2 - \frac{1}{2}(1-2\theta)\mu_4$ with μ_2 an eigenvalue of D^2.

For eigenvalues of R_D, we have

$$-\tau^{-2}D^2 Y_1 + \tfrac{1}{2}(1-2\theta) D^4 Y_1 + \tau^{-2}D^2 Y_2 = \lambda Y_1,$$
$$\tau^{-2}D^2 Y_1 - \tau^{-2}D^2 Y_2 + \tfrac{1}{2}(1-2\theta) D^4 Y_2 = \lambda Y_2.$$

A similar argument yields that the eigenvalues and eigenfunctions of R_D are

$$\{\tfrac{1}{2}(1-2\theta)\mu_4, (Y, Y)\},$$
$$\{-2\tau^{-2}\mu_2 + \tfrac{1}{2}(1-2\theta)\mu_4, (Y, -Y)\}.$$

Because R_D and Q_D have a common set of eigenfunctions, the inequality $R_\tau \leqslant CQ\tau$ is equivalent to

$$\lambda(R_D) \leqslant C\lambda(Q_D) \tag{3.8}$$

for the corresponding eigenvalues.

It follows from Fourier analysis that the eigenvalues of the operators D^2 and D^4 are

$$\mu_2 = 2h^{-2}(\cos 2\pi jh - 1), \qquad \mu_4 = \mu_2^2, \qquad j = 1, 2, ..., J.$$

Thus we have

$$\lambda_j^R = -2\tau^{-2} \cdot 2h^{-2}(\cos 2\pi jh - 1) + (1-2\theta)\, 2h^{-4}(\cos 2\pi jh - 1)^2,$$
$$\lambda_j^Q = -2\tau^{-2} \cdot 2h^{-2}(\cos 2\pi jh - 1) - (1-2\theta)\, 2h^{-4}(\cos 2\pi jh - 1)^2.$$

It follows from (3.8) that

$$4\tau^{-2} \cdot h^{-2}(1-\cos 2\pi jh) + 2(1-2\theta)\, h^{-4}(1-\cos 2\pi jh)^2 \leqslant C[4\tau^{-2} \cdot h^{-2}(1-\cos 2\pi jh) - 2(1-2\theta)\, h^{-4}(1-\cos 2\pi jh)^2];$$

i.e.,

$$2 + (1-2\theta)\gamma^2(1-\cos 2\pi jh) \leqslant C[2 - (1-2\theta)\gamma^2(1-\cos 2\pi jh)]. \tag{3.9}$$

The inequality (3.9) holds with $C_1 = (2 + (1-2\theta)\gamma^2)/(2-(1-2\theta)\gamma^2)$, provided $\gamma < \sqrt{1/(1-2\theta)}$. This completes the proof.

LEMMA 3. *Assume that* $\gamma = (\tau/h) < \sqrt{1/(1-2\theta)}$, $0 \leqslant \theta \leqslant \frac{1}{2}$. *Then the following estimates hold*:

$$\|E^n\|_2 \leqslant C, \qquad \|E_x^n\|_2 \leqslant C, \qquad \|E^n\|_\infty \leqslant C,$$
$$\|\mathcal{N}^n\|_2 \leqslant C, \qquad \|U^{n+1/2}\|_\infty \leqslant C, \qquad \|U_x^{n+1/2}\|_2 \leqslant C,$$
$$0 \leqslant n \leqslant \tau/h.$$

Proof. It follows from Lemma 1 that

$$\|E^n\|_2 \leqslant C$$

and

$$\|E_x^{n+1}\|_2^2 + \|E_x^n\|_2^2 + \|U_x^{n+1/2}\|_2^2 + (1-2\theta)\, h \sum_{j=1}^{J} \mathcal{N}_j^{n+1} \cdot \mathcal{N}_j^n + \theta(\|\mathcal{N}^{n+1}\|_2^2 + \|\mathcal{N}^n\|_2^2) + \tfrac{1}{2}h \sum_{j=1}^{J} (\mathcal{N}_j^{n+1} + \mathcal{N}_j^n) \times (|E_j^{n+1}|^2 + |E_j^n|^2) = \text{const.}$$

Using Lemma 2, we have

$$\|E_x^{n+1}\|_2^2+\|E_x^n\|_2^2+\frac{1}{C_1}\|U_x^{n+1/2}\|_2^2 + \left(\frac{1-2\theta}{2C_1}+\theta\right)(\|\mathcal{N}^{n+1}\|_2^2+\|\mathcal{N}^n\|_2^2) + \frac{1}{2}h\sum_{j=1}^{J}(\mathcal{N}_j^{n+1}+\mathcal{N}_j^n) \times(|E_j^{n+1}|^2+|E_j^n|^2)\leqslant C. \tag{3.10}$$

The last term $\mathscr{L}$ in the above formula is estimated by

$$|\mathscr{L}|\leqslant\frac{1}{2}h\sum_{j=1}^{J}[\mathcal{N}_j^n\cdot|E_j^n|^2+\mathcal{N}_j^{n+1}\cdot|E_j^n|^2 + \mathcal{N}_j^n\cdot|E_j^{n+1}|^2+\mathcal{N}_j^{n+1}\cdot|E_j^{n+1}|^2] \leqslant\frac{\varepsilon_1}{2}h\sum_{j=1}^{J}[(\mathcal{N}_j^n)^2+(\mathcal{N}_j^{n+1})^2] + \frac{1}{2\varepsilon_1}h\sum_{j=1}^{J}[|E_j^n|^4+|E_j^{n+1}|^4], \tag{3.11}$$

for any $\varepsilon_1>0$. By the discrete Sobolev embedded theorem,

$$h\sum_{j=1}^{J}|E_j^n|^4\leqslant\|E^n\|_\infty^2\cdot h\sum_{j=1}^{J}|E_j^n|^2 \leqslant C\|E^n\|_\infty^2 \leqslant\varepsilon_2\|E_x^n\|_2^2+\frac{C}{\varepsilon_2}\|E^n\|_2^2 \tag{3.12}$$

for any $\varepsilon_2>0$.

Substituting (3.11) and (3.12) into (3.10) and choosing $\varepsilon_1=\varepsilon_2=(1-2\theta)/2C_1+\theta$, we obtain the inequality

$$\frac{1}{2}\|E_x^{n+1}\|_2^2+\frac{1}{2}\|E_x^n\|_2^2+\frac{1}{C_1}\|U_x^{n+1/2}\|_2^2 + \frac{1}{2}\left(\frac{1-2\theta}{2C_1}+\theta\right)(\|\mathcal{N}^{n+1}\|_2^2+\|\mathcal{N}^n\|_2^2)\leqslant C.$$

From the above, the following estimates can be obtained:

$$\|E_x^n\|_2\leqslant C,\qquad \|\mathcal{N}^n\|_2^2\leqslant C,\qquad \|U_x^{n+1/2}\|_2\leqslant C.$$

It follows from the discrete Sobolev embedded theorem that

$$\|E^n\|_\infty\leqslant C,\qquad \|U^{n+1/2}\|_\infty\leqslant C,$$

which are the desired results.

Next, we define the square of energy norm for (e,η) as

$$e_H^{n+1/2}=\|e_x^{n+1}\|_2^2+\|e_x^n\|_2^2+\|U_x^{n+1/2}\|_2^2 + \tfrac{1}{2}(\|\eta^{n+1}\|_2^2+\|\eta^n\|_2^2),$$

where

$$e_j^n=E(x_j,t^n)-E_j^n,\qquad \eta_j^n=\mathcal{N}(x_j,t^n)-\mathcal{N}_j^n,$$
$$(U_j^{n+1/2})_{x\bar{x}}=(\eta_j^n)_t.$$

THEOREM 1 (Convergence). *Assume that* $\gamma=(\tau/h)<\sqrt{1/(1-2\theta)}$, $0\leqslant\theta\leqslant\frac{1}{2}$, *and* $E(x,t)\in C^{(4,3)}$, $\mathcal{N}(x,t)\in C^{(4,4)}$ *for the solution of problem* (1.1)–(1.4). *Then the solution of the difference problem* (2.1)–(2.7) *converges to the solution of the problem* (1.1)–(1.4) *in the energy norm; i.e., the square of the energy norm satisfies*

$$e_H^{n+1/2}\leqslant C\cdot h^2,\qquad as\quad h\to 0.$$

The theorem can be proved in the same way as that used to prove Theorem 2 of [6]. However, for $\theta\neq\frac{1}{2}$, modification must be made to estimate the term associated with $h\sum_{j=1}^{J}\eta_j^{n+1}\cdot\eta_j^n$, which is one of the major contributions of this paper. The term associated with $h\sum_{j=1}^{J}\eta_j^{n+1}\cdot\eta_j^n$ can be estimated by using the inequality in Lemma 2 of this section, and the rest of the proof will proceed similarly as [6]. The details are omitted.

4. NUMERICAL EXPERIMENTS

We consider scheme (2.1)–(2.2) and compare $\theta=0$ with $\theta=\frac{1}{2}$ by numerical experiments. In addition, two explicit schemes for Eq. (1.1) will also be considered. The four schemes to be considered in this section are defined as

Scheme I: (2.1)–(2.2) with $\theta=\frac{1}{2}$;

Scheme II: (2.1)–(2.2) with $\theta=0$;

Scheme III: (2.2) and

$$i(E_j^{n+1})_{\bar{t}}+\tfrac{3}{2}(E_j^n)_{x\bar{x}}-\tfrac{1}{2}(E_j^{n-1})_{\bar{x}x} =\tfrac{1}{4}(\mathcal{N}_j^{n+1}+\mathcal{N}_j^n)(E_j^{n+1}+E_j^n); \tag{4.1}$$

Scheme IV: (2.2) and

$$i\left(\frac{E_j^{n+1}-E_j^{n-1}}{\tau}\right)+\frac{1}{h^2}(E_{j+1}^n-E_j^{n+1}-E_j^{n-1}+E_{j-1}^n) =\frac{1}{4}(\mathcal{N}_j^{n+1}+\mathcal{N}_j^n)(E_j^{n+1}+E_j^n). \tag{4.2}$$

If the exact solution of the differential equations is substituted into the difference schemes and the Taylor expansion is used, we obtain the following truncation errors:

$$\mathcal{N}_{tt}-\mathcal{N}_{xx}=(|E|^2)_{xx}+\left[-\frac{\tau^2}{12}\mathcal{N}_{tttt}+\theta\frac{\tau^2}{4}\mathcal{N}_{xxtt}+\frac{h^2}{12}\mathcal{N}_{xxxx}+\frac{h^2}{12}(|E|^2)_{xxxx}\right]+O(h^4+\tau^4+h^2\tau^2)\quad \text{for (2.2);} \tag{4.3}$$

$$iE_t+E_{xx}=\mathcal{N}E+\left[-\frac{i\tau^2}{12}E_{ttt}-\frac{\tau^2}{8}E_{xxtt}-\frac{h^2}{12}E_{xxxx}+\frac{\tau^2}{8}E\mathcal{N}_{tt}+\frac{\tau^2}{8}\mathcal{N}E_{tt}\right]+O(h^4+\tau^4+h^2\tau^2)\quad \text{for (2.1);} \tag{4.4}$$

$$iE_t+E_{xx}=\mathcal{N}E+\left[-\frac{i\tau^2}{12}E_{ttt}+\frac{3\tau^2}{8}E_{xxtt}-\frac{h^2}{12}E_{xxxx}+\frac{\tau^2}{8}E\mathcal{N}_{tt}+\frac{\tau^2}{8}\mathcal{N}E_{tt}\right]+O(h^4+\tau^2+h^2\tau^2)\quad \text{for (4.1);} \tag{4.5}$$

$$iE_t+E_{xx}=\mathcal{N}E+\left[\frac{\tau^2}{h^2}E_{xx}-\frac{i\tau^2}{3}E_{ttt}-\frac{h^2}{12}E_{xxxx}+\frac{\tau^2}{8}E\mathcal{N}_{tt}+\frac{\tau^2}{8}\mathcal{N}E_{tt}\right]+O(h^4+\tau^4)\quad \text{for (4.2).} \tag{4.6}$$

The truncation errors for Schemes I, II, and III are $O(h^2+\tau^2)$ and the truncation error for Scheme IV is $O((\tau/h)^2+h^2+\tau^2)$. Thus, $\tau=o(h)$ is required in Scheme IV.

The convergence for Schemes I and II has been proved in Section 3. Linearized stability analysis demonstrates that Scheme I is unconditionally stable and Scheme II is stable for $\tau<h$. The stability of Schemes III and IV depends on θ. They are unconditionally stable if $\theta=\frac{1}{2}$ and stable for $\tau<h$ if $\theta=0$.

The analytic solution of Zakharov equations, which is derived in [5, 9], will be used in our computation for comparison. The solution can be written as

$$E(x,t)=F(x-vt)\exp[i\varphi(x-ut)], \tag{4.7}$$

$$\mathcal{N}(x,t)=G(x-vt), \tag{4.8}$$

where

$$F(x-vt)=E_{\max}\cdot \mathrm{dn}(w,q),$$

$$G(x-vt)=\frac{|F(x-vt)|^2}{v^2-1}+\mathcal{N}_0,$$

$$w=\frac{E_{\max}}{\sqrt{2(1-v^2)}}\cdot(x-vt),$$

$$q=\frac{\sqrt{E_{\max}^2-E_{\min}^2}}{E_{\max}},$$

$$\varphi=v/2,$$

and $\mathrm{dn}(w,q)$ is a Jacobian elliptic function [7]. The analytic solution is a solitary wave. The parameters are chosen as

$$L=20,\quad E_{\max}=1,\quad E_{\min}=4.5147\times10^{-6},$$
$$v=0.628319,\quad u=-1.73692,\quad \mathcal{N}_0=0.181786.$$

Next, we will discuss the computational results. Let $T_L\approx L/v$ denote the time during which the solitary wave travels through a period and

$$WE=\max_{1\leqslant j\leqslant J}|E(x_j,t^n)-E_j^n|,$$

$$W\mathcal{N}=\max_{1\leqslant j\leqslant J}|\mathcal{N}(x_j,t^n)-\mathcal{N}_j^n|.$$

All computations were carried out on the SUN 3/80 workstation.

4.1. *Approximation for the Initial Condition*

A simple approximation for the initial condition $\mathcal{N}_t(x,0)=\mathcal{N}^1(x)$ is

$$\mathcal{N}_j^1=\mathcal{N}_j^0+\tau\cdot\mathcal{N}^1(x_j).$$

This approximation has a truncation error of order $O(\tau)$ and is not consistent with the difference schemes because the truncation errors for Schemes I, II, III are of order $O(h^2+\tau^2)$. Instead, we will use the following approximation with the truncation error $O(\tau^2)$:

$$\mathcal{N}_j^1=\mathcal{N}_j^0+\tau\cdot\left(\frac{\partial\mathcal{N}}{\partial t}\right)_{t=0}+\frac{\tau^2}{2}\cdot\left(\frac{\partial^2\mathcal{N}}{\partial^2 t}\right)_{t=0}+O(\tau^3).$$

Using the differential equation (1.2), the above formula is approximated as

$$\mathcal{N}_j^1=\mathcal{N}_j^0+\tau\cdot\mathcal{N}^1(x_j)+\frac{\tau^2}{2}[(\mathcal{N}_j^0)_{x\bar{x}}+(|E_j^0|^2)_{x\bar{x}}].$$

Numerical results obtained by using the discrete initial conditions (2.6) and (2.7) are compared in Table I. The results show that condition (2.7) is more accurate than (2.6) and the difference of errors for E produced by two initial approximations remain the same during the entire computation. For example, for $h=\tau=0.5$ in Scheme I, the

TABLE I

Comparison between Discrete Initial Conditions (2.6) and (2.7)

Step sizes	Scheme	Initial condition	Time	$WE \times 100$	$W\mathcal{N} \times 100$
$h=\tau=0.05$	I	(2.6)	8	1.91	1.42
			16	3.21	1.64
			24	4.41	2.32
			$T_L=31.85$	6.05	1.82
		(2.7)	8	1.38	0.70
			16	2.71	1.03
			24	4.08	1.38
			$T_L=31.85$	5.39	1.70
	II	(2.6)	8	1.64	0.92
			16	2.69	0.98
			24	3.62	1.29
			$T_L=31.85$	5.04	0.78
		(2.7)	8	1.11	0.16
			16	2.21	0.24
			24	3.33	0.32
			$T_L=31.85$	4.41	0.38
$h=\tau=0.2$	I	(2.6)	8	8.58	15.01
			16	14.16	21.27
			24	19.51	28.39
			$T_L=32$	26.37	33.35
		(2.7)	8	6.65	12.60
			16	12.06	19.16
			24	18.27	25.98
			$T_L=32$	24.03	32.56
	II	(2.6)	8	4.08	5.34
			16	5.73	5.89
			24	6.86	8.36
			$T_L=32$	9.91	6.84
		(2.7)	8	1.89	2.57
			16	3.64	3.75
			24	5.64	5.12
			$T_L=32$	7.41	6.24

difference of errors between (2.6) and (2.7) equals $1.91-1.38=0.53$, at $t=8$, and $6.05-5.39=0.66$, at $t=T_L=31.85$. Considering the computational expenses for both initial conditions are small in computation, formula (2.7) is preferred. Therefore, we will use formula (2.7) for computing the initial values in the subsequent computations.

4.2. *Stability for Scheme II at $\tau=h$*

In Section 3, we have proved that Scheme II is convergent if $\tau<h$. Linearized stability analysis gives a necessary condition for stability by the von Neumann condition, i.e.,

$$\tau \leqslant h,$$

but a sufficient condition is $\tau<h$. If the term $(|E_j^0|^2)_{x\bar{x}}$ is ignored, then scheme (2.2) with $\theta=0$ is unstable at $\tau=h$, since there is an unbounded solution [10]. However, one often takes $\tau=h$ in practical computations. The reason is that a slow growth of errors would probably be harmless in practice.

For Scheme II, we make the comparison of $\tau=h$ with $\tau<h$. The results are given in Table II and Figs. 1 and 2. It is clear from Table II and Figs. 1 and 2 that choosing $\tau=h$ is better than $\tau<h$ in terms of both computational time and accuracy. Furthermore, Scheme II with $\tau=h$ is found to be stable in the computation.

4.3. *Choice of Step Size of Time for Scheme I*

Scheme I is proved to be convergent unconditionally. It is also unconditionally stable in view of linearized stability analysis. Therefore, choosing the time step size τ is determined solely by the accuracy requirement and the computational time. In general, $\tau \geqslant h$ is preferred. In Table III we give the comparison for various values of τ.

It is easy to see that choosing $\tau=h$ is better than $\tau>h$, both in accuracy and computational time. In Case B, $\tau=4h$

TABLE II

Error Comparison for Scheme II

Spatial step size	Time step size	Time	$WE \times 100$	$W\mathcal{N} \times 100$
$h=0.05$	$\tau=0.05$	8	1.11	0.16
		16	2.21	0.24
		24	3.33	0.32
		$T_L=31.85$	4.41	0.38
	$\tau=0.045$	8	1.13	0.17
		16	2.25	0.26
		24	3.39	0.35
		$T_L=31.85$	4.48	0.42
$h=0.1$	$\tau=0.1$	8	1.26	0.63
		16	2.48	0.92
		24	3.77	1.27
		$T_L=31.9$	4.98	1.55
	$\tau=0.095$	8	1.32	0.66
		16	2.59	0.98
		24	3.94	1.37
		$T_L=31.92$	5.18	1.62
$h=0.2$	$\tau=0.2$	8	1.89	2.57
		16	3.64	3.75
		24	5.64	5.12
		$T_L=32$	7.41	6.24
	$\tau=0.198$	8	1.92	2.62
		16	3.73	3.86
		24	5.75	5.36
		$T_L=31.878$	7.53	6.43
	$\tau=0.16$	8	2.53	2.99
		16	4.88	4.50
		24	7.50	6.02
		$T_L=31.84$	9.93	7.71

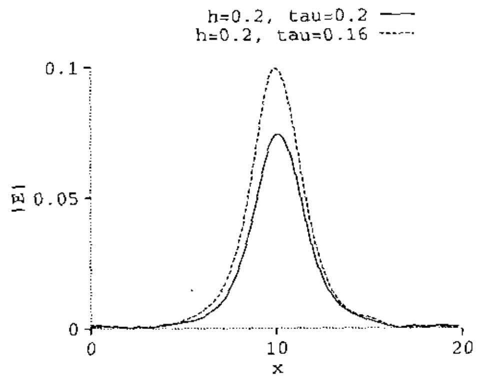

FIG. 1. Error in $|E|$ for Scheme II at $t = T_L$: A comparison for $h = \tau = 0.2$ and $h = 0.2$, $\tau = 0.16$.

is used, and its errors are much greater than the ones for $\tau = h = 0.1$. Therefore, the time step size τ should be chosen to equal h, although the scheme is stable for any τ.

4.4. *Comparison between Scheme I and Scheme II*

The numerical results obtained for Scheme I and Scheme II are compared in Table IV. The results in Table IV demonstrate that Scheme II is better than Scheme I for the step sizes used, especially for large step sizes. The CPU times for Scheme II are about 90% of those for Scheme I, when the same step size is used for both schemes. On the other side, computational errors produced by Scheme II are less than those by Scheme I and the difference between errors of the two schemes increase as the step size increases. These results may be justified by the truncation error analysis in Eq. (4.3). For Scheme II, the term $\theta(\tau^2/4)\mathcal{N}_{xxtt}$ is the truncation error (4.3) is eliminated because $\theta = 0$. In practical computations one often requires that computational errors are less than a certain tolerance at a fixed time t. To produce the same accuracy, Scheme I requires a smaller step size than Scheme II. For example, if we require that computational errors are less than 0.1 during the entire time period of the solitary wave, then the step sizes $h = \tau = 0.25$ can be taken for Scheme II, but the step sizes $h =$

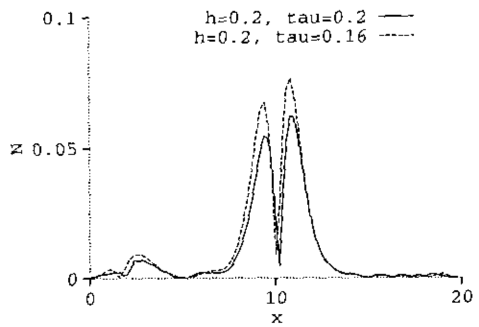

FIG. 2. Error in N for Scheme II at $t = T_L$: A comparison for $h = \tau = 0.2$ and $h = 0.2$, $\tau = 0.16$.

TABLE III

Comparison for Various τ in Scheme I

Case	Step sizes	$WE \times 100$ at T_L	$W\mathcal{N} \times 100$ at T_L	CPU time (s)
A	$h = 0.05$, $\tau = 0.1$	7.61	5.51	1067.86
	$h = \tau = 0.08$	7.54	4.66	824.21
B	$h = 0.05$, $\tau = 0.2$	15.11	21.84	537.32
	$h = \tau = 0.1$	9.45	7.41	515.78
C	$h = 0.1$, $\tau = 0.2$	16.89	23.93	266.40
	$h = \tau = 0.16$	16.88	19.86	208.30

TABLE IV

Comparison between Scheme I and Scheme II

Step sizes	Scheme	Time	$WE \times 100$	$W\mathcal{N} \times 100$	P_h^n	$H_h^{n+1/2}$	CPU time
$h = \tau = 0.05$	I	8	1.38	0.70	2.20039	2.2652	2141.04
		16	2.71	1.03	2.20039	2.2660	
		24	4.08	1.38	2.20039	2.2669	
		$T_L = 31.85$	5.39	1.70	2.20039	2.2671	
	II	8	1.11	0.16	2.20039	2.26222	1987.57
		16	2.21	0.24	2.20039	2.26226	
		24	3.33	0.32	2.20039	2.26232	
		$T_L = 31.85$	4.41	0.38	2.20039	2.26234	
$h = \tau = 0.1$	I	8	2.44	2.85	2.20039	2.26598	534.94
		16	4.68	4.34	2.20039	2.26596	
		24	7.13	5.88	2.20039	2.26592	
		$T_L = 31.9$	9.45	7.41	2.20040	2.26590	
	II	8	1.26	0.63	2.20039	2.26076	492.46
		16	2.48	0.92	2.20039	2.26076	
		24	3.77	1.27	2.20039	2.26076	
		$T_L = 31.9$	4.98	1.55	2.20039	2.26075	
$h = \tau = 0.2$	I	8	6.65	12.60	2.20039	2.27603	132.38
		16	12.06	19.16	2.20039	2.27602	
		24	18.27	25.98	2.20039	2.27599	
		$T_L = 32$	24.03	32.56	2.20040	2.27599	
	II	8	1.89	2.57	2.20039	2.25494	119.84
		16	3.64	3.75	2.20039	2.25494	
		24	5.64	5.12	2.20039	2.25494	
		$T_L = 32$	7.41	6.24	2.20039	2.25493	
$h = \tau = 0.25$	I	8	9.67	22.22	2.20039	2.28374	85.96
		16	17.25	32.05	2.20039	2.28373	
		24	25.46	42.86	2.20039	2.28372	
		$T_L = 32$	32.82	53.26	2.20039	2.28373	
	II	8	2.44	4.02	2.20039	2.25063	78.26
		16	4.63	5.91	2.20039	2.25063	
		24	7.03	8.11	2.20039	2.25062	
		$T_L = 32$	9.14	9.88	2.20039	2.25063	

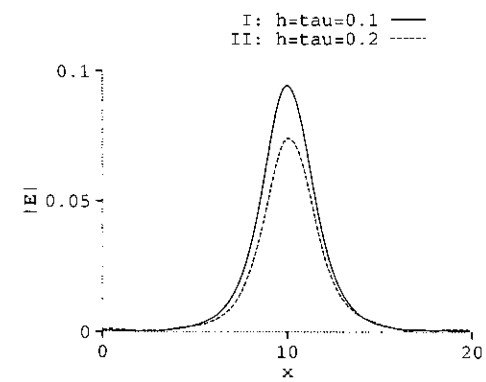

FIG. 3. Error in $|E|$ at $t = T_L$: A comparison for Scheme I with $h = \tau = 0.1$ and Scheme II with $h = \tau = 0.2$.

$\tau = 0.1$ are needed for Scheme I. To achieve the same accuracy, Scheme I takes longer CPU time than Scheme II, and the ratio of the CPU times used by the two schemes is

$$R_t = \frac{534.94}{78.26} \approx 6.8.$$

In Figs. 3 and 4, we give errors produced by Scheme I with $h = \tau = 0.1$ and Scheme II with $h = \tau = 0.2$. It is shown from these figures that the solution from Scheme II is more accurate than that from Scheme I. The rato of CPU times is

$$R_t = \frac{534.94}{119.84} \approx 4.46.$$

For small step sizes, we compare Scheme I with $h = \tau = 0.05$ and Scheme II with $h = \tau = 0.1$. The errors in the former are also more than those in the latter, and the ratio of CPU times is

$$R_t = \frac{2141.04}{492.46} \approx 4.35.$$

These results demonstrate that Scheme II is more accurate and more efficient than Scheme I.

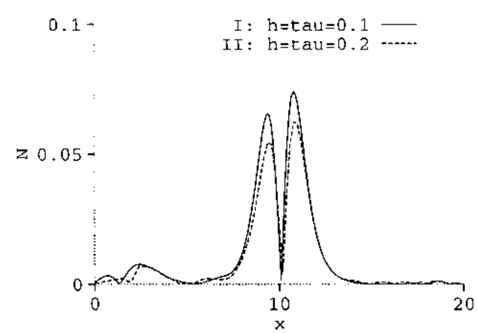

FIG. 4. Error in N at $t = T_L$: A comparison for Scheme I with $h = \tau = 0.1$ and Scheme II with $h = \tau = 0.2$.

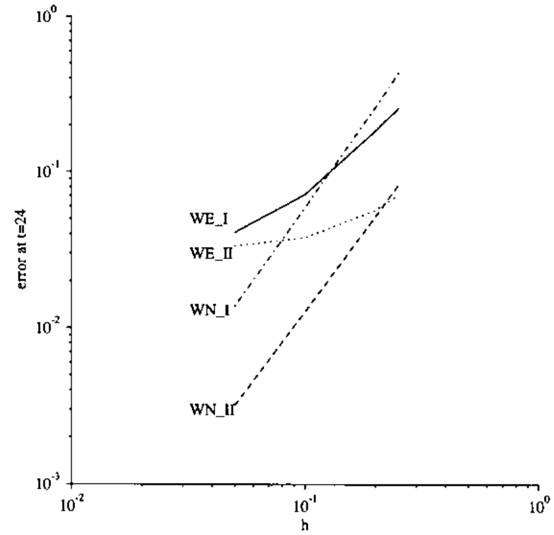

FIG. 5. Rates of convergence for Scheme I and Scheme II.

At this point, we discuss rates of convergence measured from the numerical results. The rate of convergence can be defined as

$$r = \frac{\log(E(h_1)) - \log(E(h_2))}{\log(h_1) - \log(h_2)},$$

where $E(h)$ is the measured error using step size h. From Table IV, the measured error in $\mathcal{N}$ at $t = 24$ for Scheme I is 42.86 when $h = \tau = 0.25$ and is reduced to 1.38 when $h = \tau = 0.05$. This yields an average rate of convergence of $r_I^{\mathcal{N}} = 2.1$. For Scheme II, we find $r_{II}^{\mathcal{N}} = 2.0$. Similarly, the average rates of convergence computed from errors in E at $t = 24$ are $r_I^E = 1.1$ and $r_{II}^E = 0.5$ for Scheme I and Scheme II, respectively. The rate of convergence can also be observed from Fig. 5 where the errors are displayed on a log–log scale.

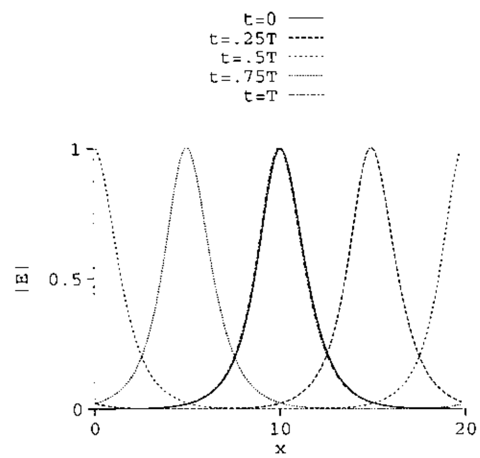

FIG. 6. $|E|$ computed by Scheme II with $h = \tau = 0.2$.

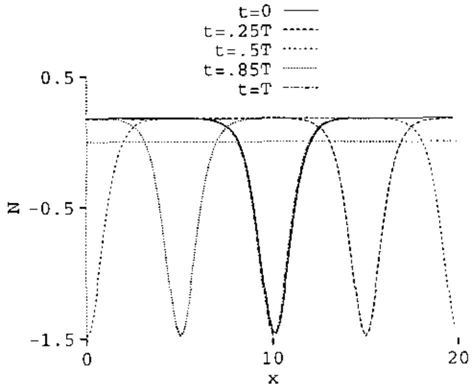

FIG. 7. N computed by Scheme II with $h = \tau = 0.2$.

From Fig. 5, we observe that in both Scheme I and Scheme II, the rate of convergence of $\mathcal{N}$ is approximately a constant (≈ 2). However, the rate of convergence of E is not a constant; the rate decreases as the step size h decreases. Furthermore, the rate of convergence is faster in Scheme I than in Scheme II. This phenomenon has not been satisfactorily explained from our analysis.

In Figs. 6 and 7, the solitary waves at various times computed by Scheme II with $h = \tau = 0.2$ are given. The wave at $t = T_L$ agrees with the one at $t = 0$ quite well. This also demonstrates the accuracy of Scheme II. It is shown from the Table IV that both Schemes I and II possess satisfactory conservative property.

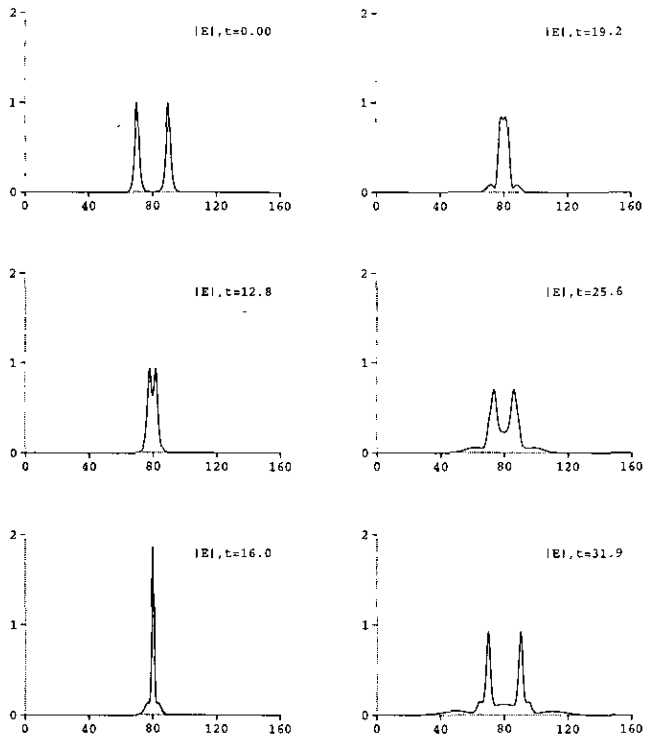

FIG. 8. $|E|$ computed by Scheme II with $h = \tau = 0.25$: Two colliding solitons at various t.

4.5. *Instability for Scheme III*

A property of Scheme III is that a linearized Crank–Nicolson scheme is used in the parabolic-type equation (1.1). The linearized scheme is explicit and its truncation error is $O(h^2 + \tau^2)$. The linearized C–N scheme is stable for many linear differential equations and the nonlinear Schrödinger equation. However, it follows from numerical

TABLE V

Results for Scheme III and $h = 0.1$, $\tau = 0.005$

θ	n	t	WE	$W\mathcal{N}$	P_h^n	$H_h^{n+1/2}$
0.5	10	0.05	1.86×10^{-3}	1.44×10^{-4}	2.20039	2.26376
	12	0.06	1.46×10^{-2}	2.28×10^{-4}	2.20043	2.38098
	14	0.07	0.12	8.08×10^{-4}	2.20268	10.42
	16	0.08	0.96	5.66×10^{-3}	2.35842	575.54
	18	0.09	7.87	3.92×10^{-2}	13.38	41166.4
	20	0.10	64.94	0.65	810.03	3.007×10^6
0	10	0.05	1.85×10^{-3}	1.39×10^{-4}	2.20039	2.26359
	12	0.06	1.45×10^{-2}	2.28×10^{-4}	2.20043	2.38085
	14	0.07	0.12	8.03×10^{-4}	2.20268	10.42
	16	0.08	0.96	5.66×10^{-3}	2.35862	576.34
	18	0.09	7.87	3.92×10^{-2}	13.40	41227.1
	20	0.10	64.95	0.65	811.23	3.011×10^6

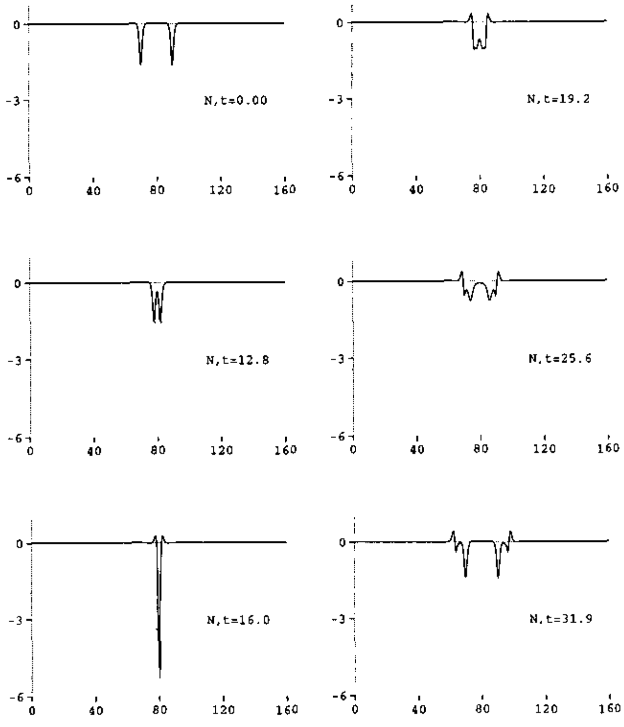

FIG. 9. N computed by Scheme II with $h = \tau = 0.25$: Two colliding solitons at various t.

TABLE VI

Errors for Scheme IV with $\theta = 0$

Step sizes				
h	τ	Time	WE	$W\mathcal{N}$
0.05	0.01	8	1.64	0.99
		16	0.63	1.13
		24	1.45	1.46
		$T_L = 31.85$	1.08	1.32
0.1	0.01	8	1.64	1.01
		16	0.63	1.18
		24	1.39	1.53
		$T_L = 31.83$	1.01	1.39
	0.05	8	1.66	0.86
		16	0.74	0.72
		24	1.53	0.58
		$T_L = 31.85$	1.49	0.59
	0.095	8.075	1.37	0.82
		16.15	1.16	1.48
		24.22	0.93	1.77
		$T_L = 31.92$	1.05	1.86
	0.1	8	1.15	0.71
		16	1.33	3.48
		24	171.17	645366
		$T_L = 31.9$	329.6	4.5×10^6
0.2	0.1	8	1.65	0.81
		16	0.85	0.60
		24	1.35	0.38
		$T_L = 31.9$	1.71	0.31

computation that Scheme III is unstable for the Zakharov equations. The computational results are given in Table V. It is clear that the solution for Scheme III, especially E, tends to infinity very steeply and an instability is produced in computing E.

4.6. *Errors for Scheme IV*

We know from (4.6) that the truncation error for the Scheme IV is $O(\tau^2/h^2)$ and this is too large for computation. Computational results for Scheme IV are given in Table VI. It follows from the results that Scheme IV is stable if $\tau < h$ and the instability is caused by computing $\mathcal{N}$ with $\tau = h$. Computational errors are always large in various step sizes and it is difficult to choose step sizes to reduce the errors.

4.7. *Collision of Two Solitary Waves*

Finally, we compute the collision of two solitary waves. For this case, two solitary waves with the same value of $E_{\max} = 1.0$ but with oppositely directed velocities are considered. The period $L = 160.0$ is chosen. Other parameters are given as

$$E_{\max} = 1.0, \qquad E_{\min} = 1.0535 \times 10^{-31}, \qquad v = \pm 0.628319,$$
$$u = \mp 2.24323, \qquad \mathcal{N}_0 = 0.0227232.$$

Schemes I and II are used to compute the collision of the two solitary waves. The computational results again demonstrate that Scheme II is better than Scheme I in both computational accuracy and CPU time. The collision procedure of two solitary waves computed by Scheme II with $h = \tau = 0.25$ are given Figs. 8 and 9. These figures roughly show the collision procedure, although the step sizes are large. In the computation, the invariants $P_h^n = 4.40078$ and $H_h^{n+1/2} = 4.6971$ are conserved. In conclusion, the semi-explicit scheme, Scheme II, with the initial approximation (2.7) is the most efficient and accurate of the four difference schemes.

ACKNOWLEDGMENT

The authors thank the referees for their comments leading to the improvement of this paper.

REFERENCES

1. Q. Chang, *Sci. Sinica Ser. A* **26**, 687 (1983).
2. Q. Chang and G. Wang, *J. Comput. Phys.* **88**, 362 (1990).
3. Q. Chang, G. Wang, and B. Guo, *J. Comput. Phys.* **93**, 360 (1991).
4. Q. Chang and L. Xu, *J. Comput. Math.* **4**, 191 (1986).
5. R. Glassey, *J. Comput. Phys.* **100**, 377 (1992).
6. R. Glassey, *Math. Comput.* **58**, 83 (1992).
7. W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for Special Functions of Mathematical Physics* (Springer-Verlag, New York, 1966).
8. T. Ortega and J. M. Sanz-Serna, *Numer. Math.* **58**, 215 (1990).
9. G. L. Payne, D. R. Nicholson, and R. M. Downie, *J. Comput. Phys.* **50**, 482 (1983).
10. R. Richtmyer and K. Morton, *Difference Methods for Initial-Value Problems* (Interscience, New York, 1967).
11. C. Sulem and P. L. Sulem, *C.R. Acad. Sci. Paris Ser. A, Math.* **289**, 173 (1979).
12. V. E. Zakharov, *Sov. Phys. JETP* **35**, 908 (1972).